Relation between the probability density and other properties of a stationary random process

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We consider the Pope-Ching differential equation [Phys. Fluids A **5**, 1529 (1993)] connecting the probability density $p_x(x)$ of a stationary, homogeneous stochastic process $x(t)$ and the conditional moments of its squared velocity and acceleration. We show that the solution of the Pope-Ching equation can be expressed as $n(x)\langle|v(x)|^{-1}\rangle$, where $n(x)$ is the mean number of crossings of the x level per unit time and $\langle|v(x)|^{-1}\rangle$ is the mean inverse velocity of crossing. This result shows that the probability density at x is fully determined by a one-point measurement of crossing velocities, and does not imply knowledge of the $x(t)$ behavior outside of the infinitesimally narrow window near x. [S1063-651X(99)06709-4]

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The relationship between the probability density functions (PDF) and the conditional means of a stationary random process has recently attracted much attention, mostly in connection with the Pope-Ching equation (PCE), Refs. [1,2], connecting the PDF $p(x)$ of the random process (RP) $x(t)$ with the conditional means of its acceleration and squared velocity. Thus, for a process with zero mean one has

$$-\frac{\partial}{\partial x}(\langle\ddot{X}|x\rangle p)+\frac{\partial^2}{\partial x^2}(\langle\dot{X}^2|x\rangle p)=0, \tag{1}$$

where $\langle\ \rangle$ denotes the conditional ensemble or the time average. The ordinary differential equation, Eq. (1), has a solution

$$p(x)=\frac{C}{\langle\dot{X}^2|x\rangle}\exp\left(\int_0^x\frac{\langle\ddot{X}|x\rangle}{\langle\dot{X}^2|x\rangle}dx\right), \tag{2}$$

where the numerical constant C is to be determined from the normalization conditions. The result, Eq. (1), was formulated in connection with probability density distributions of passive scalars advected by turbulent flow and found many applications in the experimental and theoretical work on such advection [3–16]. On the other hand, the PCE is not of statistical, but of dynamical nature (it is related to the Liouville equation) and is thus an exact result which applies not only to random processes but also to any stationary, homogeneous process like complex oscillatory phenomena or dynamical chaos. In what follows we discuss the overall behavior of the solution and its relation to the other mean values characterizing the process.

Let us first review the derivation of the PCE and discuss some simple examples. The PDF of X, $p_x(x)$, is obtained as an ensemble average (e.g., over the initial conditions) of the realizations for each of which

$$p(x,t)=\delta(X(t)-x). \tag{3}$$

The coarse-grained probability is then given by $p(x)=\langle p(x,t)\rangle$. Differentiating Eq. (3) with respect to time one gets

$$\frac{\partial p}{dt}=-\dot{X}\frac{\partial p}{\partial x}=-\frac{\partial}{\partial x}(\dot{X}p) \tag{4}$$

since X is not dependent on x. Note that Eq. (4) is a Liouville equation, and the derivation here is parallel to one given in Ref. [17]. Applying the same procedure for the second time we get

$$\frac{\partial^2 p}{dt^2}=-\frac{\partial}{\partial x}\frac{\partial}{\partial t}(\dot{X}p)=\frac{\partial}{\partial x}(\ddot{X}p)+\frac{\partial}{\partial x}\left[\dot{X}\frac{\partial}{\partial x}(\dot{X}p)\right]$$
$$=-\frac{\partial}{\partial x}(\ddot{X}p)+\frac{\partial^2}{\partial x^2}(\dot{X}^2p), \tag{5}$$

which is, of course, the same exact equation as the Liouville equation. The PCE follows after ensemble averaging, under which, for a stationary process, the time derivative vanishes, and the conditional means appear instead of the instantaneous velocity and acceleration, so that Eq. (5) reduces to Eq. (1).

Let us apply Eqs. (1) and (2) to a simple dynamical process, for example, to a harmonic oscillation, $X(t,\varphi)=\sin(\omega t+\varphi)$, for which $\langle\ddot{X}|x\rangle=a(x)=-\omega^2x$ and $\langle\dot{X}^2|x\rangle=v^2(x)=\omega^2(1-x^2)$. In this case no averaging over the initial phase φ is necessary. The solution of the PCE then reads $p(x)=C/(\omega^2\sqrt{1-x^2})$, which after normalization gives us the known result, $p_x(x)=1/(\pi\sqrt{1-x^2})$. This result corresponds to $p_x(x)=2/T|v(x)|$, where $v(x)$ is the instantaneous velocity at x, which leads to another interpretation of $p_x(x)$: this probability is proportional to the amount of time spent in the vicinity of x. As we proceed to show, $p(x)$ is always given by some special mean value of inverse velocity at point x. The more complex dynamical examples [e.g., $X(t)=\cos t+\cos 2t$, where the phase portrait of the process consists for $x<0$ of two branches crossing the line $X(t)=x$ with different velocities] shed light on the interpretation of conditional means. One can prove that the correct definition in-

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cludes weighing the values of squared velocity and of acceleration on each branch with the corresponding inverse velocity values, i.e., with the times spent in the vicinity of the *x*-line crossing.

Let us turn to a general situation. The definition $\langle F(x,t)p(x,t)\rangle = \langle F(x,t)|x\rangle p_x(x)$ is based on the Bayesian formula

$$p(x,t) = p(t|x)p_x(x). \qquad (6)$$

Here $p(x,t)$ is a joint probability distribution of the values of t and x of the measurement time and position [to make it clear imagine a trajectory of the process on the (t,x) plane; the probability to pick up a measurement result (t,x) corresponds to a δ ridge following a line given by the equation of motion]. Considering our measurement as lasting for the time T we can define this as $p(x,t) = (1/T)\,\delta(X(t)-x)$, which is an explicit function of x and an implicit function of t. On the other hand, one can consider another Bayesian expression, $p(x,t) = p(x|t)p_t(t)$, with the time-sampling probability $p_t(t) = 1/T$ and the conditional probability $p(x|t) = \delta(X(t)-t)$. To get $p_x(x)$ from Eq. (6) we need to express $p(x,t)$ and $p(t|x)$ as *explicit functions of t* and implicit functions of *x*, i.e., to change variables in the corresponding probability distributions. Doing this one is led to the following expressions: For $p(x,t)$ one gets

$$p(x,t) = \sum_i \frac{1}{|v_i(x)|}\,\delta(t-T_i(x)),$$

where i numbers the roots of the equation $X(t)=x$ on the real axis between 0 and T, i.e., the *x*-line crossings. The conditional density $p(t|x)$ is given by

$$p(t|x) = \left(\sum_i \frac{1}{|v_i(x)|}\right)^{-1} \sum_i \frac{1}{|v_i(x)|}\,\delta(t-T_i(x)).$$

This means that for a dynamical process the value of $\langle v^2|x\rangle$ is not equal to $\langle v^2(x)\rangle$ but is given by

$$\langle v^2|x\rangle = \left(\sum_i \frac{1}{|v_i(x)|}\right)^{-1} \sum_i |v_i(x)|.$$

As a parallel,

$$\langle a|x\rangle = \left(\sum_i \frac{1}{|v_i(x)|}\right)^{-1} \sum_i \frac{a(x)}{|v_i(x)|}$$

gives us the corresponding acceleration. The probability distribution $p_x(x)$ then reads

$$p_x(x) = \frac{1}{T}\sum_i \frac{1}{|v_i(x)|}.$$

This probability density satisfies the Pope-Ching equation, which can be checked by substitution:

$$-\left(\sum_i \frac{1}{|v_i(x)|}\right)^{-1} \sum_i \frac{a(x)}{|v_i(x)|}\left(\frac{1}{T}\sum_i \frac{1}{|v_i(x)|}\right) + \frac{d}{dx}\left(\sum_i \frac{1}{|v_i(x)|}\right)^{-1} \sum_i |v_i(x)|\left(\frac{1}{T}\sum_i \frac{1}{|v_i(x)|}\right) = 0$$

since for each branch or realization i one has $d|v_i(x)|/dx = a(x)/|v_i(x)|$, since $[dv(x)/dx]dx = a(x)(dt/dx)dx = [a(x)/v(x)]dx$. The averaging over the ensemble of realizations gives

$$p_x(x) = \left\langle \frac{1}{T}\sum_i \frac{1}{|v_i(x)|}\right\rangle,$$

which reduces in the limit of $T\to\infty$ and for stationary processes to

$$p_x(x) = n(x)\left\langle \frac{1}{|v(x)|}\right\rangle. \qquad (7)$$

Here $n(x)$ is the mean number of *x*-level crossings per unit time and the average is taken over the values of the velocity at crossing. In this average each crossing (branch) is counted with the same weight.

At first glance, Eq. (7) seems to make no sense, since, for example, for a Gaussian process $p_v(v)$ has a maximum at $v=0$ and thus delivers a divergence of $p_x(x)$. This interpretation is wrong, since $v(x)$ is a velocity measured *provided* $x(t)$ has just crossed an observation line. Note that a crossing cannot take place with zero velocity. (This last statement is essentially the content of Bulinskaya's theorem, see p. 76 of Ref. [18].)

Imagine we have a long run of data points giving the $x(t)$ values with a high sampling frequency τ^{-1}. The velocity $v(x)$ is measured if the process $x(t)$ has an *x*-level crossing in the interval $t, t+\tau$. Let $p(v,x)$ be the joint probability density of the distribution of (v,x). Then, for τ small, $x(t)$ obeys the inequality $x - v\tau < x(t) < x$ if $x(t)$ crosses the x line from below. The probability that this happens (i.e., the probability that x falls into the interval considered) is

$$\int_{x-v\tau}^{x} p(v,x)\,dx = |v|p(v,x). \qquad (8)$$

The same expression is obtained when considering the *x*-level crossing from above. This leads to the conclusion that the velocity distribution at the crossing is proportional to $p(v(x)) \propto |v|p(v,x)$. Normalizing this we get

$$p(v(x)) \propto |v|p(v,x) \Big/ \int_{-\infty}^{\infty} |v|p(v,x)\,dv. \qquad (9)$$

On the other hand, the denominator in Eq. (9) is just the expression defining the density of *x*-level crossings, which, according to Rice's formula, Refs. [18,19], is given by $n(x) = \int_{-\infty}^{\infty}|v|p(v,x)\,dv$, see also Ref. [20]. Thus the mean inverse velocity at the crossing multiplied by the mean crossing density is

$$n(x)\langle|v(x)|^{-1}\rangle \propto \int_{-\infty}^{\infty} \frac{1}{|v(x)|}|v(x)|p(v,x)dv$$
$$= \int_{-\infty}^{\infty} p(v,x)dv \equiv p_x(x), \qquad (10)$$

which delivers another, purely statistical way of deriving Eq. (7).

Let us discuss in more detail the difference between the two types of averages involved in the PCE and in Eq. (7). Let us fix the time-sampling window τ. The values of velocities and accelerations are then determined from the data points x_i by taking $v_i=\tau^{-1}(x_{i+1}-x_i)$ and $a_i=\tau^{-2}(x_{i+1}-2x_i+x_{i-1})$. The procedure leading to conditional means corresponds to the averaging over the set of data for which $x_i\in[x-\Delta/2,x+\Delta/2]$, with Δ being a width of the x-sampling window. If many subsequent points occur within the window, all of them count. On the other hand, the crossing mean does not suppose any observation window and is defined by averaging over the set of data for which $x_i<x$ and $x_{i+1}>x$, i.e., for the pairs of x points that definitely correspond to crossing the x level.

Equation (7) has a very interesting statistical implication: it means that in order to obtain the value of the probability density of a stationary, homogeneous random process at x, one does not need to perform the continuous measurement of x or to keep the sampled track of the process. The one-point probability density is a truly local characteristic of the process and is fully determined by a one-point measurement of crossing velocities. It does not imply the knowledge of the processes behavior outside of the infinitesimally narrow window near x. This is of extreme interest when we really are interested in the behavior of RP in some narrow domain of x. Note that both the standard sampling procedure (building a frequency histogram of the process) and also obtaining the probability density with the help of PCE through the evaluation of the conditional mean values of squared velocity and of acceleration need a full range of knowledge about the process, since otherwise the normalization constant cannot be obtained. On the other hand, Eq. (7) corresponds to a very simple time averaging, in which the overall sum of inverse measured crossing velocities during the observation time T is divided by T.

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